

Tau-functions for Quiver Gauge Theories

A. Marshakov

Lebedev Institute, ITEP and NRU HSE, Moscow, Russia

e-mail: mars@lpi.ru, mars@itep.ru

Abstract

The prepotentials for the quiver supersymmetric gauge theories are defined as quasi-classical tau-functions, depending on two different sets of variables: the parameters of the UV gauge theory or the bare complexified couplings, and the vacuum condensates of the theory in IR. The bare couplings are introduced as degenerate periods on the UV base curve, and the consistency of corresponding gradient formulas for the tau-functions is proven using the Riemann bilinear relations. It is shown, that dependence of generalised prepotentials for the quiver gauge theories upon the bare couplings turns to coincide with the corresponding gradient formulas for the tau-functions of the isomonodromic deformations. Computations for the $SU(2)$ quiver gauge theories with bi- and tri-fundamental matter are performed explicitly and analysed in the context of 4d/2d correspondence.

1 Introduction

It has been understood since [1] that quiver theories with the semisimple gauge groups, consisting of the product of several independent factors, lead to important new insight on geometry of $\mathcal{N} = 2$ supersymmetric gauge theories. In particular they uncover few issues related to the moduli spaces of the base curves, corresponding to gauge theories in the ultra-violet (UV). Being previously observed only in higher perturbations of the UV prepotential [2, 3, 4, 5], this geometry becomes nontrivial for quiver theories and can be studied in detail.

For generic quivers the problem is quite complicated technically, though can be always reformulated in terms of flat connections on base UV curves [6, 7]. However, it can be described, as usual, in common language of integrable systems [8], and this can be rather instructive for their further study. The important low-energy part of the story for the infra-red (IR) effective theory is contained in the Seiberg-Witten (SW) prepotential [9], which is a particular case of the Krichever tau-function [10], if not being restricted to dependence on the IR condensates. The extension to quasiclassical tau-function becomes necessary to gain the UV information in the final answer as well, e.g. if one considers a (super)conformal gauge theory. Below we are going to show, that dependence on UV couplings can be considered for the quiver tau-functions almost on equal footing with the dependence upon IR condensates: both are encoded by the period integrals and satisfy integrability constraints, coming from the Riemann bilinear relations (RBR) on the SW cover Σ of the UV curve Σ_0 . The essential difference comes however from the fact, that all ingredients of the tau-function construction concerning the UV information can be projected to the base curve Σ_0 , while “pure IR” characteristics remain directly related to the properties of the cover Σ .

The starting point for the quiver gauge theory is UV base curve $\Sigma_0 = \Sigma_{g_0, n}$ of genus g_0 with n punctures, and the complex dimension of its Teichmüller space

$$L = 3g_0 - 3 + n \quad (1.1)$$

counts the number of the gauge group $G = \bigotimes_{r=1}^L G_r$ factors. For the UV-completeness each factor should have non-negative beta function, i.e.

$$\begin{aligned} \beta_r = 2N_c^{(r)} - N_f^{(r)} - \sum_{r': \langle r, r' \rangle \neq 0} N_c^{(r')} N_{bf}^{(r, r')} - \\ - \sum_{r', r'': \langle r, (r', r'') \rangle \neq 0} N_c^{(r')} N_c^{(r'')} N_{3f}^{(r, r', r'')} - \dots \geq 0, \quad r = 1, \dots, L \end{aligned} \quad (1.2)$$

This condition obviously has solutions, when restricted to fundamental and bi-fundamental matter. The well-known cases are $N_f^{(r)} \leq 2N_c^{(r)}$ with all $N_{bf}^{(r, r')} = N_{3f}^{(r, r', r'')} = \dots = 0$ (in this case quiver is “decoupled”), and, for example, conformal quivers with $N_{bf}^{(r, r')} = 2$, $N_c^{(r)} = N_c^{(r')}$, $N_{3f}^{(r, r', r'')} = \dots = 0$, while already for the tri-fundamental matter it can be hardly extended

beyond the $SU(2)$ case. In practice, for getting consistent UV formulation, it restricts the base curves to have $g_0 = 0, 1$, though the higher genera $g_0 > 1$ can be also considered formally using, for example, the language of pants-decomposition of Σ_0 (see, e.g. [11, 12]). However, the latter class of theories cannot be always formulated in weak-coupling regime [1, 13] (and presumably does not have generally a Lagrangian description), and this should be reflected in the problems with their geometric formulation. Fortunately, neither of these complications are present for the $SU(2)$ -quivers, when for all gauge group factors $N_c = 2$, and even some explicit computations for these theories can be performed and will be presented below.

For rational and elliptic UV curves the geometric picture is rather transparent. If $g_0 = 1$ ($n \geq 1$) there exists a canonical holomorphic 1-form $d\omega$ on Σ_0 , e.g. normalized as $\oint_{A^{(0)}} d\omega = 1$ (the base curves Σ_0 should be always taken with fixed basis of cycles in $H^1(\Sigma_0)$, i.e. over a point in the Teichmüller space). For $g_0 = 0$ one just takes a degeneration of this picture, i.e. considers a sphere with $n \geq 3$ marked points and canonical differential

$$d\omega = d\Omega_{z_n z_{n-1}}^{(0)} = \frac{dz}{z - z_n} - \frac{dz}{z - z_{n-1}} \quad (1.3)$$

($z \in \mathbb{C}$, by an $SL(2, \mathbb{C})$ -transformation one can always put $z_n = 0$ and $z_{n-1} = \infty$). However, there is no such canonical choice for $g_0 \geq 2$.

The geometry of the UV curve can be therefore described in terms of the following co-ordinates on its Teichmüller space:

- For $g_0 = 0$ ($n \geq 3$) fix 3 punctures (e.g. z_n, z_{n-1} and z_{n-2}), and consider the third kind Abelian differential (1.3). Then its $(n-3)$ “periods”

$$\begin{aligned} \log q_j &= \int_{B_j^{(0)}} d\omega = \int_{B_j^{(0)}} d\Omega_{z_n z_{n-1}}^{(0)} = \int_{z_{n-2}}^{z_j} d\Omega_{z_n z_{n-1}}^{(0)} = \\ &= \log \frac{(z_j - z_n)(z_{n-2} - z_{n-1})}{(z_j - z_{n-1})(z_{n-2} - z_n)}, \quad j = 1, \dots, n-3 \end{aligned} \quad (1.4)$$

are just the cross-ratios to be identified with the complexified bare couplings, and play the role of desired co-ordinates in what follows.

For $g_0 = 1$ ($n \geq 1$) take instead the canonical holomorphic differential $d\omega = dz$, where now $z \in \mathbb{C}/\Gamma(1, \tau_0)$ and consider

$$\begin{aligned} \tau_0 &= \oint_{B_0} d\omega = \int_0^{\tau_0} dz \\ \tau_j^{(0)} &= \int_{B_j^{(0)}} d\omega = \int_{P_n}^{P_j} d\omega = \int_0^{z_j} dz = z_j, \quad j = 1, \dots, n-1 \end{aligned} \quad (1.5)$$

with a single fixed puncture P_n , which by translation can be always put to $z(P_n) = z_n = 0$.

- On the cover Σ , defined by a polynomial equation (with the coefficients, taking values in the k -differentials on Σ_0) one defines extra period co-ordinates

$$a_I = \frac{1}{2\pi i} \oint_{A_I} dS, \quad I = 1, \dots, g \quad (1.6)$$

- The definition of the quasiclassical tau-function [10] is given by:

$$a_I^D = \frac{\partial \mathcal{F}}{\partial a_I} = \frac{1}{2\pi i} \oint_{B_I} dS, \quad I = 1, \dots, g \quad (1.7)$$

and

$$\frac{\partial \mathcal{F}}{\partial \tau_j^{(0)}} = \frac{1}{2} \int_{A_j^{(0)}} \frac{dS}{d\omega} dS \quad (1.8)$$

where dS is the SW differential for quiver gauge theory. The structure of last formula is identical to the dependence of the Krichever tau-function upon the variables, associated with the non single valued differentials in [10]. However, when the variables (1.5) are associated with degenerate periods (which always happens for $g_0 = 0$) the r.h.s. of second formula in (1.8) is reduced to the computation of residues, and its consistency can be proven using the RBR for the second-kind meromorphic Abelian differentials, see Appendix A.

- The variables (1.4), (1.5) can be considered as co-ordinates on the Teichmüller space of the base curve Σ_0 . The deformations w.r.t. their higher analogs can be introduced by similar to (1.8) gradient formulas

$$\frac{\partial \mathcal{F}}{\partial T_j^{(k)}} = \frac{1}{2k} \int_{A_j^{(0)}} \left(\frac{dS}{d\omega} \right)^{l_k} dS \quad (1.9)$$

at least for $T_j^{(k)} = 0$, $k > 0$ (and $T_j^{(1)} = \tau_j^{(0)}$) with some integers $\{l_k\}$. This formula will be discussed below for particular quiver theories.

2 Gauge quivers, complex curves and integrable models

2.1 SU(2) quivers and hyperelliptic curves

For the quiver gauge theory with gauge group $G = SU(2)^{\otimes L}$ and n (bi-)fundamental matter multiplets condition (1.2) acquires the form

$$\beta_r = 4 - N_f - 2N_{bf} - 4N_{3f} - \dots \geq 0, \quad r = 1, \dots, L \quad (2.1)$$

in each $SU(2)$ -factor. Such beta-functions vanish and correspond to the superconformal theories for $N_f = 2$, $N_{bf} = 1$, or for $N_f = 0$, $N_{bf} = 2$, when $N_{3f} = \dots = 0$. This happens for UV curves with $g_0 = 0$ if number of factors is $L = n - 3$, and for $g_0 = 1$ with $L = n$, the corresponding bare couplings are then introduced by (1.4) and (1.5) respectively.

These UV variables can be interpreted as co-ordinates on the Teichmüller space of the base curve, and to add IR condensates one has to consider the covering curve, or locally - Teichmüller deformations. To do this [1, 14] one can endow the UV curve $\Sigma_0 = \Sigma_{g_0, n}$ with a two-differential $t = t(z)dz^2$, which at vicinity of each puncture looks as

$$t(z) \underset{z \rightarrow z_j}{=} \frac{\Delta_j}{(z - z_j)^2} + \frac{u_j}{z - z_j} + \dots, \quad j = 1, \dots, n \quad (2.2)$$

where the residues Δ_j , $j = 1, \dots, n$ are fixed, and are related to the bare masses of quiver gauge theory. With the fixed residues (2.2) is defined up to a generic holomorphic two-differential

$$\delta t = \sum_{j=1}^n \delta u_j h_j + \sum_{k=1}^{3g_0-3} \delta y_k h_k = \sum_{j=1}^n \frac{\delta u_j}{z - z_j} + \text{reg} \quad (2.3)$$

so totally we have exactly (1.1) parameters of deformation, corresponding to a basis in the space of holomorphic 2-differentials $(\{h_k\}, \{h_j\})$, and in the massless case $\Delta_j = 0$ the two-differential (2.2) is holomorphic itself. This case will be analysed below in detail.

The Seiberg-Witten curve Σ in the $SU(2)$ -quiver theory covers Σ_0 twice

$$x^2 = t(z) \quad (2.4)$$

with branchings at the zeroes $t = 0$. The genus g of Σ can be counted, for example, by the Riemann-Hurwitz formula

$$2 - 2g = \#S(2 - 2g_0) - \#BP \quad (2.5)$$

where the number of sheets for the hyperelliptic curve (2.4) $\#S = 2$, and due to the Riemann-Roch theorem for the 2-differential (2.2) the number of branching points is calculated as

$$\#BP = \#(\text{zeroes } t) - \#(\text{poles } t) + 4(g_0 - 1) = 2n + 4g_0 - 4 \quad (2.6)$$

One finds therefore

$$g = 1 + 2g_0 - 2 + n + 2g_0 - 2 = 4g_0 - 3 + n = L + g_0 \quad (2.7)$$

for the full genus of the SW cover Σ .

2.2 Gaudin model

For the lower genera $g_0 = 0, 1$ the curve (2.4) can be associated with the rational and/or elliptic Gaudin model (see e.g. [15, 16, 17]). Consider the meromorphic Lax operator, for example, on sphere $g_0 = 0$ with n punctures (the elliptic generalisation is straightforward)

$$L(z) = \sum_{j=1}^n \frac{A_j}{z - z_j}, \quad A_j = \begin{pmatrix} \mu_j + h_j & e_j \\ f_j & \mu_j - h_j \end{pmatrix} \in gl_2 \quad (2.8)$$

which defines the following spectral curve equation for the double-cover Σ of the rational base curve $\Sigma_0 = \Sigma_{0,n}$

$$\det(x - L(z)) = x^2 - x \operatorname{Tr} L(z) + \det L(z) = 0 \quad (2.9)$$

After the shift $x \rightarrow x - \frac{1}{2} \operatorname{Tr} L(z) = x - \sum_{i=1}^n \frac{\mu_i}{z - z_i}$, it acquires the form of (2.4)

$$\begin{aligned} x^2 = t(z) &= -\det L(z) + \frac{1}{4} (\operatorname{Tr} L(z))^2 = \frac{1}{2} \operatorname{Tr} L(z)^2 - \frac{1}{4} (\operatorname{Tr} L(z))^2 = \\ &= \sum_{i=1}^n \frac{\frac{1}{2} \operatorname{Tr} A_i^2 - \frac{1}{4} (\operatorname{Tr} A_i)^2}{(z - z_i)^2} + \sum_{i=1}^n \frac{1}{z - z_i} \sum_{j \neq i} \frac{\operatorname{Tr}(A_i A_j) - \frac{1}{2} \operatorname{Tr} A_i \operatorname{Tr} A_j}{z_i - z_j} \end{aligned} \quad (2.10)$$

i.e. with

$$\begin{aligned} \Delta_j &= h_j^2 + e_j f_j = \frac{1}{2} \operatorname{Tr} A_j^2 - \frac{1}{4} (\operatorname{Tr} A_j)^2 \\ u_j &= \sum_{k \neq j} \frac{2h_j h_k + e_j f_k + e_k f_j}{z_j - z_k} = \sum_{j \neq i} \frac{\operatorname{Tr}(A_i A_j) - \frac{1}{2} \operatorname{Tr} A_i \operatorname{Tr} A_j}{z_i - z_j} \end{aligned} \quad (2.11)$$

$j = 1, \dots, n$

are the Casimir functions (fixed residues) and the $SL(2)$ -Gaudin integrals of motion correspondingly, satisfying additional constraints¹

$$\sum_{j=1}^n u_j = 0, \quad \sum_{j=1}^n (z_j u_j + \Delta_j) = 0, \quad \sum_{j=1}^n (z_j^2 u_j + 2z_j \Delta_j) = 0 \quad (2.12)$$

coming from the transformation properties of the 2-differential in (2.10). Differently, they can be ensured by constraining the total momentum $e = \sum_{j=1}^n e_j = 0$ and $f = \sum_{j=1}^n f_j = 0$, and that can be achieved by global $SL(2)$ -conjugation of $L(z)$, while $h = \sum_{j=1}^n h_j = 0$ corresponds to natural vanishing of the total spin projection.

Equation (2.10) exactly corresponds to the double-covering curve (2.4) of genus $g = n - 3$ for the case $g_0 = 0$. One can now introduce

$$dS = x dz = \frac{y dz}{\prod_{j=1}^n (z - z_j)} \quad (2.13)$$

¹In particular, these constraints guarantee the absence of an extra pole at $z = \infty$, which arises otherwise for the 2-differential $t(z)(dz)^2$ in (2.10), and therefore ensure correct counting in (2.6), (2.7).

where

$$\begin{aligned}
y^2 = R(z) &= \sum_{j=1}^n (-u_j z + \Delta_j + u_j z_j) \prod_{l \neq j} (z - z_l)^2 = \\
&= z^{2n-1} \sum_{j=1}^n u_j + z^{2n-2} \sum_{j=1}^n (z_j u_j + \Delta_j) + z^{2n-3} \sum_{j=1}^n (z_j^2 u_j + 2z_j \Delta_j) + R_{2n-4}(z)
\end{aligned} \tag{2.14}$$

i.e. upon (2.12) equation (2.10) turns into a more common form for a hyperelliptic curve

$$y^2 = R_{2n-4}(z) \tag{2.15}$$

of genus $g = n - 3$. It has totally $2n$ punctures, which are pairwise related by hyperelliptic involution $y \leftrightarrow -y$, and so do the residues of the generating differential $dS = xdz$

$$\text{res}_{P_j^\pm} xdz = \text{res}_{P_j^\pm} \frac{ydz}{\prod_{j=1}^n (z - z_j)} = \pm \sqrt{\Delta_j}, \quad j = 1, \dots, n \tag{2.16}$$

which define the mass-parameters in the theory up to their total sum $\sum_{j=1}^n \mu_j$, absorbed in the shift of x -variable in (2.10). In what follows it would be convenient using $SL(2, \mathbf{C})$ -transformations to fix in (2.13) three points to be $(0, 1, \infty)$, i.e. to consider

$$dS = xdz = \frac{ydz}{z(z-1) \prod_{j=1}^{n-3} (z - q_j)} \tag{2.17}$$

when one of the points in (2.15) is moved to infinity, giving rise to an odd power polynomial in the r.h.s. Similarly one can treat elliptic case with $g_0 = 1$, see e.g. [15].

2.3 $SU(N)$ generalisation

The technique of the Gaudin models can be applied to the case of $SU(N)$ quivers, though the results already for the curves are much less explicit and practically useful. The gl_N -valued analog of the Lax operator (2.8)

$$L(z) = \sum_{j=1}^n \frac{A_j}{z - z_j}, \quad A_j \in gl_N \tag{2.18}$$

gives rise to the polynomial spectral curve equation

$$\det(x - L(z)) = x^N + \sum_{k=2}^N (-)^k t_k(z) x^{N-k} = 0 \tag{2.19}$$

with the coefficients

$$\begin{aligned}
t_k(z) &= \text{Tr}(\underbrace{L(z) \wedge \dots \wedge L(z)}_k) = \sum_{j_1 \dots j_k} \frac{\text{Tr}(A_{j_1} \wedge \dots \wedge A_{j_k})}{(z - z_{j_1}) \dots (z - z_{j_k})} = \\
&= \sum_{j=1}^n \sum_{l=1}^k \frac{u_{l,j}^{(k)}}{(z - z_j)^l}, \quad k = 2, \dots, N
\end{aligned} \tag{2.20}$$

to be naturally interpreted as k -differentials on the UV curve Σ_0 , where again the total trace can be absorbed by shift $x \rightarrow x - \frac{1}{N} \text{Tr} L(z) = x - \sum_{j=1}^N \frac{\mu_j}{z - z_j}$ so that all matrices in (2.20) after redefinition $A_j \rightarrow A_j - \mu_j \mathbf{1}$ become sl_N -valued.

The total number of coefficients in (2.19) is $n \sum_{k=2}^N k = n \left(\frac{N(N+1)}{2} - 1 \right)$. Among them there are $n(N-1)$ Casimir functions $\{u_{k,j}^{(k)}\} = \{K_j^{(k)}\}$, $j = 1, \dots, n$, $k = 2, \dots, N$, and, up to the global group action, one finally gets

$$n \left(\frac{N(N+1)}{2} - 1 \right) - n(N-1) - N^2 + 1 = n \frac{N(N-1)}{2} - N^2 + 1 \tag{2.21}$$

integrals of motion of the $SL(N)$ Gaudin model. This number exactly equals to genus of (2.19): for example, the application of the Riemann-Hurwitz formula gives now

$$2 - 2g = \#S(2 - 2g_0) - \#BP = 2N - \#BP \tag{2.22}$$

where the number of sheets of the cover (2.19) is $\#S = N$, and the number of branching points can be easily found, say, in particular degenerate case $x^N = t_N(z)$ of (2.19). By the Riemann-Roch theorem the number of zeroes of the N -differential from (2.20) on sphere is $\#\text{poles} + 2N(g_0 - 1) = N(n - 2)$ and each such point has multiplicity $(N - 1)$ so that

$$\begin{aligned}
\#BP &= N(N-1)(n-2) \\
g &= 1 - N + \frac{1}{2} \#BP = n \frac{N(N-1)}{2} - N^2 + 1
\end{aligned} \tag{2.23}$$

coinciding finally with (2.21). Another way to get the same genus formula of the SW curve comes from the adjunction formula for the compactified curve (2.19), but we find below that this is not what we need for quiver gauge theories.

Instead of cotangent bundle to the Teichmüller space one can consider here the space of the $SL(N, \mathbb{C})$ -valued flat connections (see e.g. [7] and references therein) on base curve with complex dimension

$$\begin{aligned}
\dim_{\mathbb{C}}(\mathcal{M}_{sl_N}(\Sigma_0)) &= 2g_0(N^2 - 1) + n(N^2 - N) - 2(N^2 - 1) = \\
&\stackrel{g_0=0}{=} 2 \left(n \frac{N(N-1)}{2} - N^2 + 1 \right) = 2g
\end{aligned} \tag{2.24}$$

One finds that genus (2.23) equals to the half of maximal dimension of the phase space in the $SL(N)$ -Gaudin theory. Indeed, one can treat (2.24) as counting the number of independent parameters in the coefficients of connection (2.18) - matrices A_1, \dots, A_n , with the fixed traces $\mu_j^{(k)} = \text{Tr} A_j^k$, $k = 2, \dots, N$, subjected to $\sum_{j=1}^n A_j = 0$ (absence of the extra pole in (2.18) at $z = \infty$) and modulo conjugation by the diagonal gauge group, i.e. for rational curve $\Sigma_0 = \Sigma_{0,n}$

$$\begin{aligned} n(N^2 - 1) - n(N - 1) - 2(N^2 - 1) &= n(N^2 - N) - 2(N^2 - 1) = \\ &= \dim_{\mathbb{C}}(\mathcal{M}_{sl_N}(\Sigma_0))|_{g_0=0,n} \end{aligned} \quad (2.25)$$

For the higher-rank $SU(N)$ quiver theories one can also have “special punctures” [1] in the marked points of $\Sigma_{0,n}$. This corresponds to the $SU(r)$ flavour symmetry at each puncture or “smaller orbits” described by the rank r matrices $A_j = \mathbf{a}_j^\dagger \otimes \mathbf{b}_j$ for $j = 1, \dots, n - 2$. In the simplest case with $r = 1$ one gets the minimal orbits of dimension $2(N - 1)$ ($2N$ components with vanishing $(a^\dagger, b) = 0$ taken modulo $U(1)$ flavour group action). In such case the maximal dimension (2.25) is reduced by $\Delta_{n,N} = (n - 2)(N(N - 1) - 2(N - 1)) = (n - 2)(N - 1)(N - 2)$, i.e. becomes

$$\begin{aligned} \dim_{\mathbb{C}}(\mathcal{M}_{sl_N}(\Sigma_0))|_{g_0=0,n} - \Delta_{n,N} &= n(N^2 - N) - 2(N^2 - 1) - (n - 2)(N - 1)(N - 2) = \\ &= 2(N - 1)(n - 3) \equiv 2\tilde{g} \end{aligned} \quad (2.26)$$

and grows only linearly with N . It is easy to see, that $\tilde{g} = (N - 1)(n - 3)$ is indeed the genus of the cover in the reduced $r = 1$ case, and this nicely fits with expression (2.7) for $g_0 = 0$, when the dimension of orbit is always as in the $r = 1$ case.

For the coefficients (2.20) in the spectral curve equation (2.19) one finds, that for rank r matrices in (2.18) at j -th puncture, the maximal order of the pole does not exceed r . For example, when all (except for two special marked points [1], say z_n and z_{n-1}) matrices A_j for $j = 1, \dots, n - 2$ are of unit rank $A_j \wedge A_j = 0$, and this restricts all poles at z_1, \dots, z_{n-2} to be maximally first order.

Under such constraints one gets in (2.20)

$$u_{k,j}^{(l)} = 0, \quad j = 1, \dots, n - 2, \quad l = 2, \dots, k \quad (2.27)$$

It means also, that counting the number of branching points and the result for smooth genus \tilde{g} of the cover $\tilde{\Sigma}$ is now different: instead of (2.23) one gets

$$\begin{aligned} \widetilde{\#BP} &= 2(N - 1)(n - 2) \\ \tilde{g} &= 1 - N + \frac{1}{2}\widetilde{\#BP} = (N - 1)(n - 3) \end{aligned} \quad (2.28)$$

which exactly fits with (2.26) in the same sense as before. The number of branch points $\widetilde{\#BP}$, leading to the second line by means of the Riemann-Hurwitz formula, can be computed again in particular case, when all k -differentials vanish in (2.19) except for $t_N(z)$. Then all branching

points come with multiplicity $N - 1$ for N -cover of Σ_0 , and their total number is $2(n - 1)$, coming from the $n - 1$ first-order poles at z_j , $j = 1, \dots, n - 2$ and from $n - 1$ zeroes of

$$t_N(z) = \frac{Q_{n-1}(z)(dz)^N}{(z - z_n)^N(z - z_{n-1})^N \prod_{j=1}^{n-1}(z - z_j)} \quad (2.29)$$

$$\#(\text{zeroes } t_N) - \#(\text{poles } t_N) = (n - 1) - (2N + n - 1) = -2N = 2N(g_0 - 1)$$

for some polynomial $Q_{n-1}(z)$.

3 Prepotentials and tau-functions

Let us now turn directly to the tau-functions [10] for the $SU(2)$ quiver gauge theories. The SW periods (1.6) are defined for (2.10) in a standard way by

$$a_I = \frac{1}{2\pi i} \oint_{A_I} x dz, \quad I = 1, \dots, g \quad (3.1)$$

and, additionally, one has to add the degenerate periods (1.4), corresponding to the bare couplings. For the rational UV curve $g_0 = 0$ and fixed $(z_{n-2}, z_{n-1}, z_n) = (1, \infty, 0)$ we introduce

$$i\pi\tau_j^{(0)} = \int_{B_j^{(0)}} \frac{dz}{z} = \int_1^{q_j} \frac{dz}{z} = \log q_j, \quad j = 1, \dots, n - 3 \quad (3.2)$$

and the definition of the extended prepotential $\mathcal{F} = \mathcal{F}(\mathbf{a}, \mathbf{q})$ for quiver theory should be also completed by the gradient formulas (1.8).

The derivatives of so defined tau-functions over the bare couplings

$$q_j \frac{\partial \mathcal{F}}{\partial q_j} = \frac{1}{2} \oint_{A_j^{(0)}} \frac{dS}{d\omega} dS = \frac{1}{2} \oint_{A_j^{(0)}} x^2 z dz, \quad j = 1, \dots, n \quad (3.3)$$

are expressed for the variables (3.2) through the integrals over the dual cycles to $\{B_j^{(0)}\}$ on base curve, or the residues

$$q_j \frac{\partial \mathcal{F}}{\partial q_j} = \frac{1}{2} \text{res}_{q_j} \frac{dS}{d\omega} dS = \frac{1}{2} \text{res}_{q_j} x^2 z dz = \frac{1}{2} (\Delta_j + u_j q_j), \quad j = 1, \dots, n - 3 \quad (3.4)$$

and integrability condition is satisfied, due to $\frac{\partial u_i}{\partial q_j} = \frac{\partial u_j}{\partial q_i}$. This becomes a nontrivial relation, when the derivatives are taken at fixed *periods* (3.1), and this coincides with integrability conditions for the tau-functions of the isomonodromic problem, see below.

A direct way to prove the consistency of (3.3) is to use the RBR for the differentials

$$d\Omega_i = q_i \frac{\partial}{\partial q_i} dS = q_i \frac{\partial x}{\partial q_i} dz, \quad i = 1, \dots, n - 3 \quad (3.5)$$

For (2.17) these derivatives obviously give the second kind Abelian differentials with the second order poles at $z = q_i$

$$d\Omega_i \underset{z \rightarrow q_i}{=} \frac{d\xi_i}{\xi_i^2} + \dots, \quad \oint_{A_k} d\Omega_i = 0, \quad i, k = 1, \dots, n-3 \quad (3.6)$$

More strictly, in the case of nonvanishing masses (2.16) each differential (3.5) is in fact a linear combination of two second kind differentials

$$\begin{aligned} q_i \frac{\partial}{\partial q_i} dS &= d\Omega_i^{(+)} - d\Omega_i^{(-)}, \quad i = 1, \dots, n-3 \\ d\Omega_i^{\pm} \underset{z \rightarrow q_i, \pm}{=} \frac{d\xi_i^{\pm}}{(\xi_i^{\pm})^2} + \dots, \quad \oint_{A_k} d\Omega_i^{\pm} &= 0, \quad i, k = 1, \dots, n-3 \end{aligned} \quad (3.7)$$

with the only poles at $z \rightarrow q_i$ at each of the sheets of the double cover (2.4), (2.15). In the massless limit these two poles collide into a single double-pole at the ramification point with $z = q_i$.

Consistency of formulas (3.4) becomes then a consequence of RBR for the second kind Abelian differentials (see Appendix A), since they can be rewritten in more familiar way

$$q_j \frac{\partial \mathcal{F}}{\partial q_j} = \frac{1}{2} \text{res}_{q_j} \frac{dS}{d\omega} dS = \frac{1}{2} \text{res}_{q_j} \xi_j^{-1} dS, \quad j = 1, \dots, n-3 \quad (3.8)$$

where the local co-ordinate can be defined by $\xi_j^{-1} = \frac{dS}{d\omega} \big|_{z \rightarrow q_j} = xz \big|_{z \rightarrow q_j}$ (cf. with [18, 10]) in terms of a single meromorphic function $\frac{dS}{d\omega} = xz$. Similarly one can write for the formulas (1.9) with $k > 1$, where for the $SU(2)$ quivers one has to put $l_k = 2k + 1$

$$\frac{\partial \mathcal{F}}{\partial T_j^{(k)}} = \frac{1}{2k} \text{res}_{q_j} \left(\frac{dS}{d\omega} \right)^{2k+1} dS = \frac{1}{2k} \text{res}_{q_j} \xi_j^{-2k-1} dS, \quad j = 1, \dots, n-3 \quad (3.9)$$

which acquire exactly the form of the derivatives of prepotential over the parameters of UV deformation, considered in [5].

3.1 Massless case

One can get far more in explicit form, considering the limit of all vanishing flavour masses, when (3.30) turns into

$$\begin{aligned} x^2 = t(z) &= \frac{V_{n-4}(z)}{\prod_{i=1}^n (z - z_i)} \\ dS = xdz &= \sqrt{\frac{V_{n-4}(z)}{\prod_{i=1}^n (z - z_i)}} dz \rightarrow \sqrt{U} \frac{\sqrt{\prod_{j=1}^{L-1} (z - v_j)} dz}{\sqrt{z(z-1) \prod_{k=1}^L (z - q_k)}} \end{aligned} \quad (3.10)$$

where, as usual, it is convenient to fix three of the branching points in the denominator to be 0, 1 and ∞ , and use the number of gauge groups $L = n - 3$. Equation (3.10) describes itself the *holomorphic* generating differential (notice, that its variations w.r.t. U and $\{v_j\}$ are also holomorphic and span the L -dimensional space of linearly independent first kind Abelian differentials) on the hyperelliptic curve

$$Y^2 = z(z-1) \prod_{k=1}^L (z - q_k) \prod_{l=1}^{L-1} (z - v_l) \quad (3.11)$$

of genus L . It means, therefore that

$$\begin{aligned} dS = xdz &= \sum_{J=1}^L a_J d\omega_J = \frac{dz}{Y} \sqrt{U} \prod_{l=1}^{L-1} (z - v_l) \\ a_I^D &= \oint_{B_I} xdz = \sum_{J=1}^L T_{IJ} a_J \end{aligned} \quad (3.12)$$

Computing q -derivatives for the differential (3.12) at fixed a -periods, one gets

$$d\Omega_j = q_j \frac{\partial}{\partial q_j} dS = \frac{q_j \sqrt{U}}{2(z - q_j)} \frac{\sqrt{\prod_{l=1}^{L-1} (z - v_l)} dz}{\sqrt{z(z-1) \prod_{k=1}^L (z - q_k)}} + d\varpi_j, \quad j = 1, \dots, n-3 \quad (3.13)$$

where $d\varpi_j = \frac{\phi_j(z)dz}{Y}$ are holomorphic, fixed by $\oint_{A_i} d\Omega_j = 0$. At $z \rightarrow q_j$ expressions in (3.13) behave as

$$d\Omega_j \simeq \frac{dz}{2(z - q_j)^{3/2}} \frac{\sqrt{q_j U \prod_{l=1}^{L-1} (q_j - v_l)}}{\sqrt{(q_j - 1) \prod_{k=j}^L (q_j - q_k)}} \simeq \frac{d\xi_j}{\xi_j^2}, \quad j = 1, \dots, n-3 \quad (3.14)$$

where the local co-ordinates now

$$\frac{1}{\xi_j} = \left. \frac{dS}{dz/z} \right|_{z \rightarrow q_j} = \frac{1}{\sqrt{z - q_j}} \frac{\sqrt{q_j U \prod_{l=1}^{L-1} (q_j - v_l)}}{\sqrt{(q_j - 1) \prod_{k=j}^L (q_j - q_k)}}, \quad j = 1, \dots, n-3 \quad (3.15)$$

are (conveniently normalised) co-ordinates at ramification points of (3.11).

3.2 Four flavours and elliptic identities

In the simplest case of single ($L = n - 3 = 1$) $SU(2)$ -gauge theory with four massless hypermultiplets, with $H = u_2$, $q = \frac{(z_2 - z_1)(z_4 - z_3)}{(z_4 - z_2)(z_3 - z_1)}$ the differential (2.13) is just

$$dS = xdz = \frac{\sqrt{H(z_2 - z_1)(z_3 - z_2)(z_4 - z_2)} dz}{\sqrt{\prod_{j=1}^4 (z - z_j)}} \rightarrow \frac{\sqrt{Hq(q-1)} dz}{\sqrt{z(z-1)(z-q)}} \quad (3.16)$$

a holomorphic differential on torus, or a particular case of (3.12) for $L = 1$ with $U = Hq(q-1)$.

Its periods can be computed in terms of the elliptic integrals, e.g.

$$a = \frac{1}{2\pi i} \oint_A x dz = \frac{\sqrt{Hq(1-q)}}{2\pi} \oint_A \frac{dz}{\sqrt{z(z-1)(z-q)}} = \sqrt{Hq(1-q)} \alpha(\tau) \quad (3.17)$$

where the A -cycle encircles $z = 0$ and $z = q$. Using the Weierstrass uniformization with

$$q = \frac{e_2 - e_3}{e_1 - e_3} = \frac{\theta_2^4(0|\tau)}{\theta_3^4(0|\tau)} \quad (3.18)$$

one can directly express

$$\begin{aligned} \alpha(\tau) &= \frac{1}{2\pi} \oint_A \frac{dz}{\sqrt{z(z-1)(z-q)}} = \frac{\sqrt{e_1 - e_3}}{\pi} \oint_A \frac{dx}{\sqrt{4(x-e_1)(x-e_2)(x-e_3)}} = \\ &= \frac{2\omega}{\pi} \sqrt{e_1 - e_3} = \theta_3^2(0|\tau) \end{aligned} \quad (3.19)$$

in terms of the theta-constants. The computation of the dual period on torus obviously gives

$$\frac{\partial \mathcal{F}}{\partial a} = \frac{1}{2\pi i} \oint_B x dz = \frac{\sqrt{Hq(1-q)}}{2\pi} \oint_B \frac{dz}{\sqrt{z(z-1)(z-q)}} \stackrel{(3.1)}{=} a \frac{\oint_B \frac{dz}{\sqrt{z(z-1)(z-q)}}}{\oint_A \frac{dz}{\sqrt{z(z-1)(z-q)}}} = a\tau \quad (3.20)$$

and, since the modular parameter of the torus τ is here independent of a , equation (3.20) is trivially solved by $\mathcal{F} = \frac{1}{2}\tau a^2$, up to a possible tau-dependent constant. The only nontrivial relation for this prepotential is

$$\begin{aligned} q \frac{\partial \mathcal{F}}{\partial q} &= \frac{1}{2} \int_{A_0} \frac{dS}{d \log z} dS = \frac{1}{2} \int_{A_0} x^2 z dz = \\ &= \frac{1}{2} Hq(q-1) \int_{A_0} \frac{dz}{(z-1)(z-q)} = \frac{a^2}{2\alpha^2(\tau)} \int_{A_0} \frac{dz}{(z-1)(z-q)} \end{aligned} \quad (3.21)$$

where the last equality is due to (3.17). The r.h.s. is proportional to a^2 , and it means that possible tau-dependent constant actually vanishes. The remaining integral is taken over the dual to B_0 cycle, defining the only here cross-ratio (1.4)

$$\int_{B_0} \frac{dz}{z} = \int_1^q \frac{dz}{z} = \log q \quad (3.22)$$

or the UV bare coupling $\log q = i\pi\tau_0$. Hence, the A_0 -period is given by the residue (3.8), and therefore (3.21) gives rise to

$$\frac{1}{\alpha^2(\tau)} \int_{A_0} \frac{dz}{(z-1)(z-q)} = \frac{1}{\alpha^2(\tau)} \text{res}_{z=q} \frac{dz}{(z-1)(z-q)} = \frac{1}{\alpha^2(\tau)(q-1)} = \frac{d\tau}{d\tau_0} \quad (3.23)$$

Using (3.18) and (3.19), we therefore obtain

$$\frac{d\tau_0}{d\tau} = \alpha^2(\tau)(q-1) = -\frac{1}{\pi^2}\theta_3^4(0|\tau) \left(1 - \frac{\theta_2^4(0|\tau)}{\theta_3^4(0|\tau)}\right) = -\frac{1}{\pi^2}\theta_4^4(0|\tau) \quad (3.24)$$

Taking in account the identity²

$$\theta_2''(0|\tau)\theta_3(0|\tau) - \theta_3''(0|\tau)\theta_2(0|\tau) = -\theta_4(0|\tau)^4\theta_2(0|\tau)\theta_3(0|\tau) \quad (3.26)$$

one finally gets for (3.24)

$$i\pi \frac{d\tau_0}{d\tau} = \frac{1}{\pi i}\theta_4^4(0|\tau) = -\frac{1}{\pi i} \left(\frac{\theta_2''(0|\tau)}{\theta_2(0|\tau)} - \frac{\theta_3''(0|\tau)}{\theta_3(0|\tau)} \right) = 4 \frac{d}{d\tau} \log \frac{\theta_2(0|\tau)}{\theta_3(0|\tau)} \quad (3.27)$$

which is integrated to the Zamolodchikov renormalisation formula $e^{i\pi\tau_0} = q = \theta_2(0|\tau)^4/\theta_3(0|\tau)^4$ [20], see also [21, 22, 23].

In the perturbative limit $q \rightarrow 0$ one gets from these formulas

$$\log q = 4 \log \frac{\theta_2(0|\tau)}{\theta_3(0|\tau)} = i\pi\tau + \log 16 + \dots \quad (3.28)$$

where

$$\tau_{\text{pert}} = \frac{1}{i\pi} \log \frac{(a+m_1)(a+m_2)(a+m_3)(a+m_4)}{(2a)^4} \underset{m_f=0}{=} \frac{1}{i\pi} \log \frac{1}{16} \quad (3.29)$$

corresponds to finite perturbative renormalisation in massless $N_c = 2$, $N_f = 4$ superconformal theory.

3.3 Perturbative couplings for bi- and tri- fundamentals

The dependence of prepotential over condensates can be always easily recovered in perturbative limit. For the $SU(2)$ quiver gauge theory it corresponds to independent computation for each j -th factor of the quiver, in the limit $z_j \rightarrow z_n$ or $q_j \rightarrow 0$. For any fixed $1 \leq j \leq n-3$ formula (2.10) gives in such limit

$$x^2 = \frac{R_{2n-4}(z)}{(z-z_n)^2 \prod_{i \neq j}^{n-1} (z-z_i)^2} \simeq \frac{R_{2n-4}(z_j)}{(z-z_n)^2 \prod_{i \neq j}^{n-1} (z_j-z_i)^2} \quad (3.30)$$

²It can be immediately obtained by taking two derivatives of the addition formula

$$\theta_2(x+y|\tau)\theta_3(x-y|\tau)\theta_2(0|\tau)\theta_3(0|\tau) = \theta_2(x|\tau)\theta_3(x|\tau)\theta_2(y|\tau)\theta_3(y|\tau) - \theta_1(x|\tau)\theta_4(x|\tau)\theta_1(y|\tau)\theta_4(y|\tau) \quad (3.25)$$

and using $\theta_1'(0|\tau) = \theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau)$, see e.g. [19] for discussion of similar identities.

so that

$$a_j = \frac{1}{2\pi i} \oint_{A_j} x dz \simeq \text{res}_{z_j=z_n}(x dz) = \frac{\sqrt{R_{2n-4}(z_j)}}{\prod_{i \neq j}^{n-1}(z_j - z_i)} \quad (3.31)$$

$$\frac{\partial \mathcal{F}}{\partial a_j} = \frac{1}{2\pi i} \int_{B_j} x dz \simeq \frac{1}{\pi i} \frac{\sqrt{R_{2n-4}(z_j)}}{\prod_{i \neq j}^{n-1}(z_j - z_i)} \int_{z_{n-2}}^{z_j} \frac{dz}{z - z_n} \approx \frac{a_j}{\pi i} \log(z_j - z_n) \simeq a_j \tau_j^{(0)}$$

i.e. $\mathcal{F} = \frac{1}{2} a_j^2 \tau_j^{(0)} + \dots$ up to a function of the rest variables $\{a_i, \tau_i^{(0)} | i \neq j\}$, i.e. corresponding to the reduced gauge quiver. This is just a classical contribution from UV, but for the massless case (3.10) one can obtain a bit more.

For a single gauge factor the theory was solved in previous section. As a next example, consider the case with $L = 2$ or $n = 5$, which already necessarily contains an external leg, corresponding to the bi-fundamental multiplet. The generating differential (3.12)

$$dS = x dz = \frac{\sqrt{U(V-z)} dz}{\sqrt{z(z-1)(z-q_1)(z-q_2)}} \quad (3.32)$$

is holomorphic differential on hyperelliptic curve of genus $L = 2$. Consider now the perturbative degeneration, corresponding to the limits $q_1 \rightarrow 0$, $q_2 \rightarrow 1$, then (3.32) can be decomposed as

$$dS \simeq \frac{\sqrt{U(V-z)} dz}{z(z-1)} = -\sqrt{UV} \frac{\sqrt{V} dz}{z\sqrt{V-z}} + \sqrt{U(V-1)} \frac{\sqrt{V-1} dz}{(z-1)\sqrt{V-z}} =$$

$$= a_1 d\omega_1 + a_2 d\omega_2 \quad (3.33)$$

where

$$a_1 = \frac{1}{2\pi i} \oint_{A_1} dS \simeq \text{res}_{z=0} \frac{\sqrt{U(V-z)} dz}{z(z-1)} = -\sqrt{UV} \quad (3.34)$$

$$a_2 = \frac{1}{2\pi i} \oint_{A_2} dS \simeq \text{res}_{z=1} \frac{\sqrt{U(V-z)} dz}{z(z-1)} = \sqrt{U(V-1)}$$

or, in this approximation one can substitute into (3.32), (3.33) expressions for the coefficients of the curve through the periods (3.34)

$$U = a_1^2 - a_2^2, \quad V = \frac{a_1^2}{a_1^2 - a_2^2} \quad (3.35)$$

In this limit elements of the period matrix are compute the via the integrals

$$\omega_1 = \int d\omega_1 = \log \frac{\eta - 1}{\eta + 1}, \quad \eta^2 = 1 - \frac{z}{V}$$

$$\omega_2 = \int d\omega_2 = \log \frac{\eta - \sqrt{1 - \frac{1}{V}}}{\eta + \sqrt{1 - \frac{1}{V}}} \quad (3.36)$$

For example, one can check explicitly, that

$$T_{21} = \oint_{B_2} d\omega_1 \simeq \log \frac{\eta-1}{\eta+1} \Big|_{-\sqrt{1-\frac{1}{V}}}^{\sqrt{1-\frac{1}{V}}} = \log \frac{\eta - \sqrt{1-\frac{1}{V}}}{\eta + \sqrt{1-\frac{1}{V}}} \Big|_{-1}^1 \simeq \oint_{B_1} d\omega_2 = T_{12} \quad (3.37)$$

the period matrix is symmetric, and obtain for

$$\begin{aligned} \frac{1}{2\pi i} T_{11} &\simeq \frac{1}{2\pi i} \log \frac{\eta-1}{\eta+1} \Big|_{-\sqrt{1-\frac{\epsilon_1}{V}}}^{\sqrt{1-\frac{\epsilon_1}{V}}} \simeq \frac{1}{2\pi i} \log \frac{\epsilon_1^2}{16V^2} = \frac{1}{i\pi} \log \frac{4\epsilon_1(a_1-a_2)(a_1+a_2)a_1^2}{(2a_1)^4} \\ \frac{1}{2\pi i} T_{22} &\simeq \frac{1}{2\pi i} \log \frac{\eta - \sqrt{1-\frac{1}{V}}}{\eta + \sqrt{1-\frac{1}{V}}} \Big|_{-\sqrt{1-\frac{1+\epsilon_2}{V}}}^{\sqrt{1-\frac{1+\epsilon_2}{V}}} \simeq \frac{1}{2\pi i} \log \frac{\epsilon_2^2}{16(V-1)^2} = \\ &= \frac{1}{i\pi} \log \frac{4\epsilon_2(a_1-a_2)(a_1+a_2)a_2^2}{(2a_2)^4} \end{aligned} \quad (3.38)$$

It is easy to see, that upon identification $4\epsilon_1 = q_1$ and $4\epsilon_2 = 1 - q_2$ expressions (3.38) reproduce perturbative logarithmic contributions to the effective couplings of two $SU(2)$ gauge factors with the numerators in the argument of logarithms, coming from fundamental and bi-fundamental matter, and denominators - from the corresponding to each gauge factor vector multiplets. Both factors according to (2.1) are superconformal gauge theories with $N_f = 2$ and $N_{bf} = 1$, since

$$\beta_r = 4 - N_f - 2N_{bf} = 0, \quad r = 1, 2 \quad (3.39)$$

Similarly one considers the case with $L = 3$ $SU(2)$ gauge factors, when the differential (3.16) becomes

$$dS = xdz = \frac{\sqrt{U(V_1-z)(V_2-z)} dz}{\sqrt{z(z-1)(z-q_1)(z-q_2)(z-q_3)}} \quad (3.40)$$

but we immediately find here new phenomenon - in the perturbative limit of such theory with $q_1 \approx 0$, $q_2 \approx 1$ and $q_3 \approx \infty$. One first gets from (3.40)

$$\begin{aligned} a_1 &= \oint_{A_1} dS \simeq \sqrt{\frac{U}{-q_3}} \text{res}_{z=0} \frac{\sqrt{(V_1-z)(V_2-z)} dz}{z(z-1)} = -\sqrt{\frac{U}{-q_3}} \sqrt{V_1 V_2} \\ a_2 &= \oint_{A_2} dS \simeq \sqrt{\frac{U}{-q_3}} \text{res}_{z=1} \frac{\sqrt{(V_1-z)(V_2-z)} dz}{z(z-1)} = \sqrt{\frac{U}{-q_3}} \sqrt{(V_1-1)(V_2-1)} \\ a_3 &= \oint_{A_3} dS \simeq \sqrt{\frac{U}{-q_3}} \text{res}_{z=\infty} \frac{\sqrt{(V_1-z)(V_2-z)} dz}{z(z-1)} = -\sqrt{\frac{U}{-q_3}} \end{aligned} \quad (3.41)$$

which corresponds to the expansion of (3.40)

$$dS = xdz \simeq \sqrt{\frac{U}{-q_3}} \frac{-V_1 V_2 dz}{z \sqrt{(V_1 - z)(V_2 - z)}} + \sqrt{\frac{U}{-q_3}} \frac{(V_1 - 1)(V_2 - 1) dz}{(z - 1) \sqrt{(V_1 - z)(V_2 - z)}} + \sqrt{\frac{U}{-q_3}} \frac{dz}{\sqrt{(V_1 - z)(V_2 - z)}} \quad (3.42)$$

over the set of three “holomorphic” differentials on degenerate curve of genus $L = 3$. Inverting formulas (3.41), we find

$$V_1 V_2 = \frac{a_1^2}{a_3^2}, \quad V_1 + V_2 = \frac{a_1^2 - a_2^2}{a_3^2} + 1 \quad (3.43)$$

and similar to (3.38) computation gives now

$$\begin{aligned} \frac{1}{2\pi i} T_{33} &= \frac{1}{2\pi i} \oint_{B_3} d\omega_3 \simeq \\ &\simeq \frac{i}{2\pi} \log \left(z - \frac{1}{2}(V_1 + V_2) + \sqrt{(z - V_1)(z - V_2)} \right) \Big|_{(\epsilon_3^{-1}, -)}^{(\epsilon_3^{-1}, +)} \end{aligned} \quad (3.44)$$

where the integration limits correspond to the points with $z = \epsilon_3^{-1}$ on “upper” and “lower” sheets of the double cover. Hence,

$$\begin{aligned} \frac{1}{2\pi i} T_{33} &\simeq \frac{1}{2\pi i} \log \frac{\epsilon_3^2 (V_1 - V_2)^2}{16} = \\ &= \frac{1}{i\pi} \log \frac{\epsilon_3 \sqrt{(a_1 + a_3 + a_2)(a_1 + a_3 - a_2)(a_1 - a_3 + a_2)(a_1 - a_3 - a_2)}}{4a_3^4} = \\ &= \frac{1}{i\pi} \log \frac{4\epsilon_3 a_3^2 \sqrt{(a_1 + a_3 + a_2)(a_1 + a_3 - a_2)(a_1 - a_3 + a_2)(a_1 - a_3 - a_2)}}{(2a_3)^4} \end{aligned} \quad (3.45)$$

where the last expression exactly corresponds to the contribution into the corresponding effective coupling of the vector $SU(2)$ multiplet in denominator under the logarithm, while the numerator together with two fundamental multiplets contains the contribution of a half-multiplet for sicilian quiver. This case again corresponds to the superconformal gauge theory due to (2.1), since

$$\beta_r = 4 - N_f - 4N_{3f} = 0, \quad r = 1, 2, 3 \quad (3.46)$$

requires for $N_f = 2$, $N_{bf} = 0$ fractional value of $N_{3f} = \frac{1}{2}$. The a -dependence of perturbative contribution (3.45) for such half-multiplet exactly corresponds to the corresponding structure of 3-vertex in Liouville theory [24], this has been already noticed in [12]. The corresponding

(perturbative) contribution to the prepotential is

$$\begin{aligned} \mathcal{F}_{\text{pert}} = & \frac{1}{2} \sum_{i=1,2,3} \tau_i^{(0)} a_i^2 + \\ & + \frac{1}{2\pi i} \left(\sum_{\pm} (a_1 \pm a_2 \pm a_3)^2 \log(a_1 \pm a_2 \pm a_3) - \sum_{i=1,2,3} (2a_i)^2 \log(2a_i) \right) \end{aligned} \quad (3.47)$$

and prepotentials of such form were considered in somewhat different, but closely related context in [25].

In fact, for the differential (3.32) the calculations are also easy in the perturbative phase with $q_1 \rightarrow 0$ and $q_2 \rightarrow \infty$ (instead of $q_2 \rightarrow 1$). One gets then

$$\begin{aligned} dS & \simeq \sqrt{\frac{U}{-q_2}} \frac{(V-z)dz}{z\sqrt{(z-1)(V-z)}} = \\ & = \sqrt{\frac{UV}{q_2}} \frac{\sqrt{V} dz}{z\sqrt{(1-z)(V-z)}} - \sqrt{\frac{U}{q_2}} \frac{dz}{\sqrt{(z-1)(z-V)}} = a_1 d\omega_1 + a_2 d\omega_2 \end{aligned} \quad (3.48)$$

with

$$a_1 = \text{res}_{z=0} dS = \sqrt{\frac{UV}{q_2}}, \quad a_2 = \text{res}_{z=\infty} dS = -\sqrt{\frac{U}{q_2}} \quad (3.49)$$

and

$$T_{22} \simeq \oint_{B_2} \frac{dz}{\sqrt{(z-1)(z-V)}} = -\log \left(z - \frac{1}{2}(V_1 + V_2) + \sqrt{(z-V_1)(z-V_2)} \right) \Big|_{(\epsilon_2^{-1}, -)}^{(\epsilon_2^{-1}, +)} \quad (3.50)$$

so that

$$\frac{1}{2\pi i} T_{22} \simeq \frac{1}{2\pi i} \log \frac{\epsilon_2^2 (V-1)^2}{16} = \frac{1}{i\pi} \log \frac{4\epsilon_2 a_2^2 (a_1 - a_2)(a_1 + a_2)}{(2a_2)^4} \quad (3.51)$$

In this limit it is also easy to compute the coupling derivatives, for example for $L = 2$ we find from (3.33):

$$\frac{dS}{d \log z} dS = x^2 z dz = \frac{U(V-z) dz}{(z-1)(z-q_1)(z-q_2)} \quad (3.52)$$

so that

$$\begin{aligned} \text{res}_{z=q_1} \frac{dS}{d \log z} dS &= \frac{U(V-q_1)}{(q_1-1)(q_1-q_2)} \\ \text{res}_{z=q_2} \frac{dS}{d \log z} dS &= \frac{U(V-q_2)}{(q_2-1)(q_2-q_1)} \end{aligned} \quad (3.53)$$

At $q_1 \rightarrow 0$, $q_2 \rightarrow \infty$ it gives after using (3.49)

$$\begin{aligned} q_1 \frac{\partial \mathcal{F}}{\partial q_1} &\simeq \frac{1}{2} \frac{UV}{q_2} = \frac{a_1^2}{2} \\ q_2 \frac{\partial \mathcal{F}}{\partial q_2} &\simeq -\frac{1}{2} \frac{U}{q_2} = -\frac{a_2^2}{2} \end{aligned} \quad (3.54)$$

corresponding to small bare couplings $\log q_1 = i\pi\tau_1 \rightarrow 0$ and $\log \frac{1}{q_2} = i\pi\tau_2 \rightarrow 0$.

4 Discussion

Let us finally turn to discussion of some general properties of the quiver tau-functions.

4.1 Coupling-derivatives and isomonodromic problem

First, return to formulas (1.8), and look in detail at their versions (3.3), (3.4), (3.8), used above mostly for the UV curves $\Sigma_0 = \Sigma_{0,n}$ with particular choice of marked points $(z_1, \dots, z_n) = (q_1, \dots, q_{n-3}, 1, \infty, 0)$. For generic position of the marked points one can write instead of (3.4)

$$\begin{aligned} q_j \frac{\partial \mathcal{F}}{\partial q_j} &= \text{res}_{z_j} \frac{dS}{d\omega} dS = \text{res}_{z_j} \frac{x^2 dz}{d\omega/dz} = \\ &= \frac{2z_j - z_n - z_{n-1}}{z_n - z_{n-1}} \Delta_j + \frac{(z_j - z_n)(z_j - z_{n-1})}{z_n - z_{n-1}} u_j \quad j = 1, \dots, n-3 \end{aligned} \quad (4.1)$$

where the choice (1.3) for $d\omega$ is used. Introduce now

$$\tilde{\mathcal{F}}(\mathbf{q}|\mathbf{a}) = \mathcal{F}(\mathbf{a}; \mathbf{q}) - \sum_{j=1}^{n-3} \log q_j^{\Delta_j} \quad (4.2)$$

The last term in the r.h.s. depends only upon the “unphysical”, non observable in the IR theory, UV couplings and can be ignored in this sense, similarly to the $U(1)$ -factors, which often appear in the context of the AGT-correspondence [14]. One can rewrite (4.1) for the redefined prepotential (4.2), with only a little change at the r.h.s.

$$q_j \frac{\partial \tilde{\mathcal{F}}}{\partial q_j} = 2 \frac{z_j - z_n}{z_n - z_{n-1}} \Delta_j + \frac{(z_j - z_n)(z_j - z_{n-1})}{z_n - z_{n-1}} u_j, \quad j = 1, \dots, n-3 \quad (4.3)$$

Now, change the variables in the l.h.s. of (4.3), according to an obvious rule

$$\tau(z_1, \dots, z_n | \mathbf{a}) = e^{\tilde{\mathcal{F}}(f(z_1), \dots, f(z_n) | \mathbf{a})} \prod_{k=1}^n f'(z_k)^{\Delta_k} \quad (4.4)$$

using the fractional-linear Möbius transformation

$$f(z) = \frac{(z - z_n)(z_{n-2} - z_{n-1})}{(z - z_{n-1})(z_{n-2} - z_n)} \quad (4.5)$$

which is consistent with the choice of the differential (1.3) in the sense, that it is normalised as

$$\begin{aligned} f(z_n) &= 0, \quad f(z_{n-1}) = \infty, \quad f(z_{n-2}) = 1 \\ f(z_j) &= q_j, \quad j = 1, \dots, n-3 \\ \tilde{\mathcal{F}}(f(z_1), \dots, f(z_n)) &= \tilde{\mathcal{F}}(q_1, \dots, q_{n-3}; 1, \infty, 0) \end{aligned} \quad (4.6)$$

It follows then from (4.3) and (4.4), that

$$\frac{\partial}{\partial z_j} \log \tau(z_1, \dots, z_n | \mathbf{a}) = u_j = \sum_{k \neq j} \frac{\text{Tr}(A_j A_k)}{z_j - z_k}, \quad j = 1, \dots, n \quad (4.7)$$

where $u_j = u_j(z_1, \dots, z_n | \mathbf{a})$ are the functions of the marked points and the periods, which can be also presented in the form (2.11). Then formula (4.7) coincides with the corresponding formula for the tau-function of the isomonodromic problem, (see e.g. [26] and references therein, the corresponding formulas can be found in Appendix B).

The detailed discussion of relation of extended quasiclassical tau-function with the tau-function of the isomonodromic problem goes far beyond the scope of this paper. One can conjecture however, that it can acquire similar to that of [26] form, since summing up with any \mathbf{z} -independent measure $d\mu(\mathbf{a})$ we find for

$$\tau(z_1, \dots, z_n) = \int d\mu(\mathbf{a}) \tau(z_1, \dots, z_n | \mathbf{a}) \quad (4.8)$$

that, following from (4.7), one can write

$$\begin{aligned} \bar{u}_j &= \bar{u}_j(z_1, \dots, z_n) = \frac{\int d\mu(\mathbf{a}) u_j(z_1, \dots, z_n | \mathbf{a}) \tau(z_1, \dots, z_n | \mathbf{a})}{\int d\mu(\mathbf{a}) \tau(z_1, \dots, z_n | \mathbf{a})} = \\ &= \frac{\partial}{\partial z_j} \log \tau(z_1, \dots, z_n), \quad j = 1, \dots, n \end{aligned} \quad (4.9)$$

i.e. the relation (4.7) is preserved for the generalised prepotentials after taking any their linear combinations with the \mathbf{z} -independent coefficients. But this is certainly just a linear relation, which should be treated only as arising in the leading order of the quasiclassical \hbar -expansion, being a particular case of the two-parameter background deformation, while the nontrivial relation of the formulas (4.7), (4.9) with the Painleve-like equations holds for the finite values of \hbar .

4.2 Higher couplings and UV deformation of the prepotential

Let us now say few words about the formulas for “higher-Teichmüller” deformations (1.9) and (3.9). Similarly to (3.52) and (3.53) one can compute, for example, in perturbative approximation for the differential (3.32):

$$\begin{aligned} \left(\frac{dS}{d \log z} \right)^{2k+1} dS &= x^{2(k+1)} z^{2k+1} dz = \\ &= U^{k+1} \left(\frac{V - z}{(z - 1)(z - q_1)(z - q_2)} \right)^{k+1} z^k dz \end{aligned} \quad (4.10)$$

so that

$$\begin{aligned} 2k \frac{\partial \mathcal{F}}{\partial T_1^{(k)}} \Big|_{T_j^{(k)} = \delta_{k1} \tau_j^{(0)}} &= \text{res}_{z=q_1} \left(\frac{dS}{d \log z} \right)^{2k+1} dS \Big|_{q_1 \rightarrow 0} = \\ &= \left(\frac{UV}{q_2} \right)^k + O(q_1) = a_1^k + O(q_1) \\ 2k \frac{\partial \mathcal{F}}{\partial T_2^{(k)}} \Big|_{T_j^{(k)} = \delta_{k1} \tau_j^{(0)}} &= \text{res}_{z=q_2} \left(\frac{dS}{d \log z} \right)^{2k+1} dS \Big|_{q_2 \rightarrow \infty} = \\ &= \left(\frac{U}{q_2} \right)^k + O(q_2^{-1}) = a_2^k + O(q_2^{-1}) \end{aligned} \quad (4.11)$$

for all $k > 0$. One finds, that formulas (1.9) and (3.9) - at least for the case of $SU(2)$ -quivers - give rise to the polynomial deformations of the UV prepotential

$$\mathcal{F}_{UV} \rightarrow \mathcal{F}_{UV} + \sum_{j,k} T_j^{(k)} \frac{a_j^k}{2k} \quad (4.12)$$

which has been already discussed for a simple gauge group in [4, 5].

4.3 Higher rank gauge quivers

We have considered here in detail only the tau-functions for $SU(2)$ gauge quivers. For the higher rank gauge groups the situation seems to be far more complicated, in particular - when in certain regions of the moduli spaces only the strong-coupling formulation in terms of the superconformal theories is known [1] - for example, instead of the tri-fundamental matter, considered in sect. 3.3. One can hopefully get more information about such strongly-coupled theories - when studying the tau-functions and period matrices along the lines proposed in the paper. We have postponed yet the analysis of the higher genus UV curves, but it is especially interesting to get progress for this case, where for higher ranked gauge groups the approach in terms of the gravity duals was developed in [13].

The extension to the higher-rank gauge theories deserves deeper understanding of the different orbit structure in the $SL(N)$ Gaudin model at special punctures and different symplectic leaves in the moduli spaces of $SL(N, \mathbb{C})$ gauge connections on Σ_0 . Complete analysis of this case requires also the study of the higher Teichmüller spaces and corresponding deformations of the UV gauge theory, which has been tested above briefly only for the $SU(2)$ gauge quivers and for the vanishing values of the deformation parameters themselves. We plan to return to all these questions elsewhere.

Acknowledgements

I am grateful to I. Krichever, J. Maldacena, A. Morozov and A. Rosly for the very useful discussions and comments, and to the organizers of the meetings in March 2012 in Osaka and July 2012 in Trieste, when some of these results were presented. This work was partly supported by RFBR grant 11-01-00962, by joint RFBR project 12-02-92108, by the Program of Support of Scientific Schools (NSh-3349.2012.2) and by the Russian Ministry of Education under the contract 8207.

Appendix

A Riemann bilinear relations

Integrability of the gradient relations for the quasiclassical tau-functions - (1.7), (1.8), (1.9), (3.8), (3.9) and similar - follow from the Riemann bilinear relations for Abelian differentials [10], which come out of the equality

$$\int_{\Sigma} d\Omega_1 \wedge d\Omega_2 = 0 \quad (\text{A.1})$$

for any two meromorphic differentials $d\Omega_1$ and $d\Omega_2$. The simplest ones, which ensure consistency of definition for the SW prepotentials (1.7), follow from (A.1) for any two canonical first class Abelian or holomorphic differentials $d\omega_I = \frac{\partial}{\partial a_I} dS$, $\oint_{A_I} d\omega_J = \delta_{IJ}$, since applying the Stokes formula on the cut Riemann surface

$$\begin{aligned} 0 &= \int_{\Sigma_g} d\omega_I \wedge d\omega_J = \int_{\partial\Sigma_g} \omega_I d\omega_J = \sum_{K=1}^g \left(\oint_{A_K} d\omega_I \oint_{B_K} d\omega_J - \oint_{A_K} d\omega_J \oint_{B_K} d\omega_I \right) = \\ &= T_{IJ} - T_{JI} \end{aligned} \quad (\text{A.2})$$

gives rise immediately to the symmetricity of the period matrix $T_{IJ} = \oint_{B_I} d\omega_J = \frac{\partial a_I^P}{\partial a_J}$.

For the non-single valued differentials similar arguments work, though derivation requires more efforts and uses integration by parts, see for example discussion of this issue in [27]. The non-single valued differentials often arise together with the formulas of the type (1.8), (1.9) when taking derivatives over the periods (1.4), (1.5) - as in the case when jumps arise after differentiating a periodic function over the period. However, after rewriting (1.9) in the form of (3.8), the integrability condition follows already from the RBR for the second-kind Abelian differentials (3.6), which are just slight modification of (A.2).

For the meromorphic differentials identity (A.1) holds on the Riemann surface Σ with punctures, and the boundary of the cut surface $\partial\Sigma$ must be supplemented by the contours $A^{(0)}$, surrounding the punctures (with and their duals $B^{(0)}$). For the differentials (3.6) with the second-order poles at $L = n - 3$ punctures one gets instead of (A.2)

$$\begin{aligned} 0 &= \int_{\Sigma_{g,L}} d\Omega_i \wedge d\Omega_j = \int_{\Sigma_{g,L}} \Omega_i d\Omega_j = \\ &= \sum_{K=1}^g \left(\oint_{A_K} d\Omega_i \oint_{B_K} d\Omega_j - \oint_{A_K} d\Omega_i \oint_{B_K} d\Omega_j \right) + \sum_{p=0}^L \oint_{A_p^{(0)}} \Omega_i d\Omega_j \end{aligned} \quad (\text{A.3})$$

The first sum in the r.h.s. vanishes due to canonical normalisation $\oint_{A_K} d\Omega_i = 0$, $\forall i$ and $K = 1, \dots, g$, so that the last term gives rise to

$$\oint_{A_i^{(0)}} \Omega_i d\Omega_j + \oint_{A_j^{(0)}} \Omega_i d\Omega_j = \oint_{A_i^{(0)}} \Omega_i d\Omega_j - \oint_{A_j^{(0)}} \Omega_j d\Omega_i = 0 \quad (\text{A.4})$$

since $\oint_{A_p^{(0)}} \Omega_i d\Omega_j = 0$ for $p \neq i, j$, and in the vicinity of any puncture Abelian integrals Ω_j are single-valued so that one can integrate by parts. Each integral in (A.4) can be calculated via the residue

$$- \oint_{A_i^{(0)}} \Omega_i d\Omega_j = 2\pi i \operatorname{res}_{z_i} \xi_i^{-1} d\Omega_j = 2\pi i q_j \frac{\partial}{\partial q_j} \operatorname{res}_{P_i} \xi_i^{-1} dS \quad (\text{A.5})$$

(one can replace $\Omega_j \underset{z \rightarrow z_j}{\simeq} -\frac{1}{\xi_j} + \dots$ by its singular part since $d\Omega_j$ are regular at $z = z_j$ for $i \neq j$), and then equality (A.5) ensures consistency of the definition (3.8).

B Isomonodromic deformations

We collect here for completeness the formulas defining tau-function of the isomonodromic deformation problem for the first order $N \times N$ matrix differential equation

$$\begin{aligned} \left(\frac{\partial}{\partial z} - A(z; \{z_j\}) \right) \Phi &= 0 \\ A(z; \{z_j\}) &= \sum_{j=1}^n \frac{A_j}{z - z_j} \end{aligned} \quad (\text{B.1})$$

Consider the flat connection in $(n + 1)$ -dimensional space

$$\begin{aligned}\mathcal{A} &= Adz + \sum_{j=1}^n \mathcal{A}_j dz_j, \quad \overline{\mathcal{A}} = 0 \\ d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} &= 0\end{aligned}\tag{B.2}$$

which satisfies in components

$$\begin{aligned}\partial_z \mathcal{A}_i - \partial_i A + [A, \mathcal{A}_i] &= 0 \\ \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i + [\mathcal{A}_i, \mathcal{A}_j] &= 0 \\ i, j &= 1, \dots, n\end{aligned}\tag{B.3}$$

For the Schlesinger ansatz

$$\mathcal{A} = \sum_{j=1}^n \frac{A_j}{z - z_j} d(z - z_j)\tag{B.4}$$

i.e. $\mathcal{A}_j = -\frac{A_j}{z - z_j} dz_j$, the system (B.3) turns into

$$\begin{aligned}\partial_i A_j + \frac{[A_i, A_j]}{a_i - a_j} &= 0, \quad i \neq j \\ \partial_i A_i &= \sum_{j \neq i} \frac{[A_j, A_i]}{a_j - a_i}\end{aligned}\tag{B.5}$$

the system of isomonodromic (by definition!) deformation equations for (B.1). Due to (B.5) one can define

$$d \log \tau = \frac{1}{2} \sum_{i \neq j} \text{Tr}(A_i A_j) d \log(z_i - z_j)\tag{B.6}$$

(the one-form in the r.h.s. is closed) the differential of the tau-function, i.e.

$$\frac{\partial \log \tau}{\partial z_i} = \sum_{j \neq i} \frac{\text{Tr}(A_i A_j)}{z_i - z_j}, \quad i = 1, \dots, n\tag{B.7}$$

and integrability condition for these relations is ensured by (B.5).

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